

Tutorial 4 (Feb 12, 14)

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(Q1) Find the mass M and the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the cube

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq a; 0 \leq y \leq a; 0 \leq z \leq a\}, \text{ where } a > 0$$

with density function $\delta(x, y, z) = x^2 + y^2 + z^2$.

Sol: Mass: $M = \iiint_D \delta(x, y, z) dV = \int_0^a \int_0^a \int_0^a (x^2 + y^2 + z^2) dz dy dx$

Method 1: Direct computations.

$$= \int_0^a \int_0^a \left[(x^2 + y^2)z + \frac{z^3}{3} \right]_0^a dy dx = \int_0^a \int_0^a \left(ax^2 + ay^2 + \frac{a^3}{3} \right) dy dx$$

$$= a \int_0^a \left[\left(x^2 + \frac{a^2}{3}\right)y + \frac{y^3}{3} \right]_0^a dx = a \int_0^a \left(ax^2 + \frac{a^3}{3} + \frac{a^3}{3} \right) dx$$

$$= a^2 \left[\frac{x^3}{3} + \frac{2}{3} a^2 x \right]_0^a = a^2 \left(\frac{a^3}{3} + \frac{2}{3} \cdot a^3 \right) = a^5.$$

Method 2: Split the triple integral into a sum of triple integrals.

$$= \int_0^a \int_0^a \int_0^a x^2 dz dy dx + \int_0^a \int_0^a \int_0^a y^2 dz dy dx + \int_0^a \int_0^a \int_0^a z^2 dz dy dx = \text{I} + \text{II} + \text{III}$$

Note that $\text{I} = \left(\int_0^a dz \right) \left(\int_0^a dy \right) \left(\int_0^a x^2 dx \right) = a \cdot a \cdot \frac{a^3}{3} = \frac{a^5}{3}$

Similarly, $\text{II} = \left(\int_0^a dz \right) \left(\int_0^a y^2 dy \right) \left(\int_0^a dx \right) = a \cdot \frac{a^3}{3} \cdot a = \frac{a^5}{3}$

$$\text{III} = \left(\int_0^a z^2 dz \right) \left(\int_0^a dy \right) \left(\int_0^a dx \right) = \frac{a^3}{3} \cdot a \cdot a = \frac{a^5}{3}$$

$$\therefore M = \text{I} + \text{II} + \text{III} = \frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3} = a^5$$

Center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$, where

$$\begin{aligned} \textcircled{i} M_{yz} &= \int_0^a \int_0^a \int_0^a x(x^2+y^2+z^2) dz dy dx \\ &= \int_0^a \int_0^a \int_0^a x^3 dz dy dx + \int_0^a \int_0^a \int_0^a xy^2 dz dy dx + \int_0^a \int_0^a \int_0^a xz^2 dz dy dx \\ &= a \cdot a \cdot \frac{a^4}{4} + a \cdot \frac{a^2}{3} \cdot \frac{a^2}{2} + \frac{a^2}{3} \cdot a \cdot \frac{a^2}{2} = a^6 \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{6} \right) = \frac{7a^6}{12} \end{aligned}$$

$$\begin{aligned} \textcircled{ii} M_{xz} &= \int_0^a \int_0^a \int_0^a y(x^2+y^2+z^2) dz dy dx \\ &= \int_0^a \int_0^a \int_0^a y^3 dz dy dx + \int_0^a \int_0^a \int_0^a yx^2 dz dy dx + \int_0^a \int_0^a \int_0^a yz^2 dz dy dx \\ &= a^6 \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{6} \right) = \frac{7a^6}{12} \end{aligned}$$

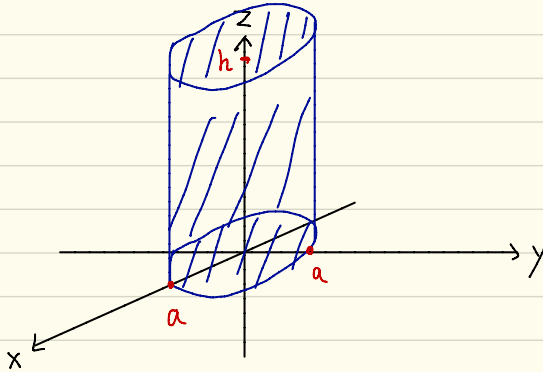
$$\textcircled{iii} M_{xy} = \int_0^a \int_0^a \int_0^a z(x^2+y^2+z^2) dz dy dx = \frac{7a^6}{12} \text{ (Similar argument as } \textcircled{i}, \textcircled{ii})$$

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\frac{7a^6}{12}}{a^5}, \frac{\frac{7a^6}{12}}{a^5}, \frac{\frac{7a^6}{12}}{a^5} \right) = \left(\frac{7a}{12}, \frac{7a}{12}, \frac{7a}{12} \right)$$

Q2) Find the moment of inertia of the solid cylinder D with radius a and height h with constant density δ about its axis.

Sol) Step 1: Parametrize D in \mathbb{R}^3 in a convenient way.

Picture:



Step 2: Describe D in cylindrical coordinates.

$$D = \{(r, \theta, z) \in [0, +\infty) \times [0, 2\pi) \times \mathbb{R} \mid 0 \leq r \leq a, 0 \leq \theta < 2\pi, 0 \leq z \leq h\}$$

Step 3: Compute the moment of inertia about the z -axis.

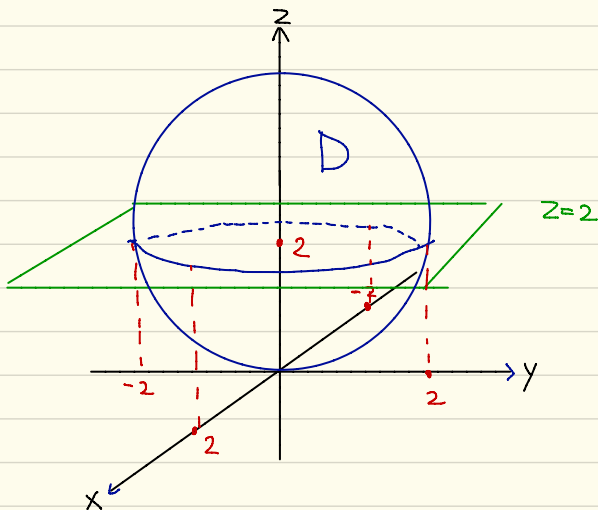
$$\begin{aligned} I_z &= \iiint_D (x^2 + y^2) \delta(x, y, z) \, dV \\ &= \int_0^h \int_0^{2\pi} \int_0^a r^2 \delta(r \cos \theta, r \sin \theta, z) (r \, dr \, d\theta \, dz) = \delta \cdot \left(\int_0^h dz \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^a r^3 \, dr \right) \\ &= \delta \cdot h \cdot 2\pi \cdot \frac{a^4}{4} = \frac{\pi}{2} \delta h a^4 \end{aligned}$$

Q3) Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{\frac{3}{2}} dz dy dx$.

Sol) Step 1: Describe the domain of integration D in \mathbb{R}^3 .

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, 2-\sqrt{4-x^2-y^2} \leq z \leq 2+\sqrt{4-x^2-y^2}\}$$

Step 2: Sketch D in \mathbb{R}^3 .



Step 3: Describe D in terms of spherical coordinates (ρ, ϕ, θ)

Note that the equation of D is $x^2+y^2+(z-2)^2 \leq 4 \Leftrightarrow x^2+y^2+z^2 \leq 4z$

In spherical coordinates: $\rho^2 \leq 4\rho \cos \phi \Leftrightarrow \rho \leq 4 \cos \phi$

$$D = \{(\rho, \phi, \theta) \in [0, +\infty) \times [0, \pi] \times [0, 2\pi) \mid 0 \leq \theta < 2\pi, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \rho \leq 4 \cos \phi\}$$

Step 4: Evaluate the integral using spherical coordinates.

$$\text{Integral} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{4\cos\phi} (\rho^2)^{\frac{2}{3}} \cdot (\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{2}} \int_0^{4\cos\phi} \rho^5 \sin\phi \, d\rho \, d\phi \right)$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{2}} \left[\frac{\rho^6}{6} \cdot \sin\phi \right]_0^{4\cos\phi} d\phi$$

$$= 2\pi \cdot \frac{4096}{6} \int_0^{\frac{\pi}{2}} \cos^6\phi \sin\phi \, d\phi$$

$$= \frac{4096}{3} \pi \left[-\frac{\cos^7\phi}{7} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{4096}{3} \pi \left(\frac{1}{7} \right) = \frac{4096}{21} \pi$$